

# Lectures: "Mini-max invariants of spaces with an action of compact Lie group"

Wacław Marzantowicz \*

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## Abstract

In this lecture we give a survey of definitions and properties of invariants which can be constructed for space  $X$  with an actions of compact Lie group and which estimate the number of critical orbits of a  $G$ -invariant function  $f : X \rightarrow \mathbb{R}$  provided  $X$  is a manifold. The already known and used in analysis are the Lusternik-Schnirelman  $G$ -category,  $G$ -genus,  $G$ -capacity, or many types of so called  $G$ -indices used in analysis, called the equivariant mini-max invariant. The common feature of all of them is the fact that to estimate of the value of each of them an appropriate version of the Borsuk-Ulam theorem is necessary. A presentation of the corresponding versions of the Borsuk-Ulam theorem, as well as some prospects and questions arisen from problems of non-linear analysis is also assumed.

*Wykład przeznaczony jest dla studentów sekcji teoretycznej i doktorantów. Przeźrocza prezentacji będą w języku angielskim, sam wykład po polsku, ale może być i po angielsku. Podane są ilości jednostek wykładowych ( $2 \times 45 \text{ min.} = 1,5 \text{ godz.}$ ) przeznaczonych na poszczególne tematy.*

*Ze względu na ograniczony czas pewne partie materiału podawane będą bez szczegółów pozostawianych do pracy własnej z literaturą. Z tego samego powodu nie będą przedstawione zastosowania do problemów analizy nieliniowej. Jednak postaramy się przedstawić aktualne problemy i ewentualnie stymulować pracę nad ich rozwiązywaniem. Sam wykład będzie próbą nowego spojrzenia na teorię, która rozwijała się dość intensywnie w latach 90-tych 20 wieku, a w ostatnim dziesięcioleciu w większości pojawiały się tylko albo jej zastosowania do bardzo specjalnych problemów nieliniowych wykorzystujące specjalne założenia analityczne.*

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# 1 Contents

- 0. Classical Lusternik-Schirelman category - 2.
  - 1. Definition and properties of the Lusternik-Schirelman category of topological space  $X$ .
  - 2. Estimate of the number of critical orbits of an invariant function by the L-S category.
  - 3. Geometrical motivation of the notion of Lusternik-Schirelman category.
  - 4. Estimate of the L-S category by the cohomological length.
- I. Equivariant Lusternik-Schirelman category - 2.
  - 1. Definition and properties of equivariant category of a  $G$ -space  $X$ .
  - 2. Estimate of the number of critical orbits of an invariant function by the equivariant category.
- II. Genus of  $G$ -space - 1.
  - 1. Definition and properties of genus of a  $G$ -space  $X$ .
  - 2. Estimates of the number of critical orbits of an invariant function by the genus.
- III. Capacity of  $G$ -space. - 1.
  - 1. Definition and properties of capacity of a  $G$ -space  $X$ .
  - 2. Estimates of the number of critical orbits of an invariant function by the capacity.
- IV. Geometrical index of  $G$ -space. - 1.
  - 1. Definition and properties of geometrical indices of a  $G$ -space  $X$ .
  - 2. Estimates of the number of critical orbits of an invariant function by the geometrical index.
- V. Cohomological index of  $G$ -space - 3.
  - 1. Definition and properties of the length of a  $G$ -space  $X$ .
  - 2. Cohomological indices of  $G$ -spaces.
  - 3. Estimates of the number of critical orbits of an invariant function by the length and cohomological index.
- VI. Computations of equivariant mini-max invariants - 3.
  - 1. Direct calculations from the definitions.
  - 2. Derivations of the values of mini-max invariants for spheres with actions of compact Lie groups by use of theorems of the Borsuk-Ulam type.
    - a) Dependence on the structure of action (representation structure) for a  $d$ -dimensional sphere with an action of a given group  $G$ .
    - b) Classifications of compact Lie groups for which the Borsuk-Ulam theorem holds for any action (representation structure) on the  $d$ -dimensional sphere.

3. Estimates of the  $G$ -category for some other  $G$ -spaces, interesting examples - e.g. a formula for  $G$ -category of a compact surface with an action of compact Lie group (in preparation).

VII. Open problems and prospects - 2.

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address: Faculty of Mathematics and Computer Sci.,  
Adam Mickiewicz University of Poznań,  
ul. Umultowska 87, 61-614 Poznań, Poland  
marzan@amu.edu.pl