

LECTURES ON POLYNOMIAL MAPPINGS

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1. INTRODUCTION

Let $g : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, for $n \geq 2$, be the germ of a holomorphic function with *isolated singularity*. This means that the Jacobian ideal:

$$\text{Jac}(g) = \left(\frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_n} \right)$$

is of finite codimension, equivalently, that the zero locus $Z(\text{Jac}(g))$ is $\{0\}$. If this zero locus is empty one says that g is *non-singular*.

Quite a lot of literature is devoted to the study of the invariants of singular functions and maps. One of the analytic invariants attached to g is its *local Milnor number* [Mi]:

$$\mu(g) = \dim_{\mathbb{C}} \mathbb{C}\{x_1, \dots, x_n\} / \text{Jac}(g).$$

It is possible to compute $\mu(g)$ from a Gröbner base. For instance, such a computation has been one of the original motivations to develop the computer algebra system Singular [GPS], which nowadays can do complicated things.

One also associates to the germ g a local fibration, called *Milnor fibration*. Then $\mu(g)$ gets a topological interpretation as the rank of the $(n - 1)$ th homology group of the Milnor fibre; it is therefore a topological invariant.

The stability of map germs (Thom, Mather, Wall, du Plessis etc) and the classification of holomorphic germs under right-equivalence or contact equivalence (Arnold and his school, Giusti, Gibson etc) are some major issues studied in the 70's and 80's. More recent topics around the topology and geometry of singularities: enumerative geometry, monodromy, relation to D-modules, Frobenius manifolds, relation to knot theory, to contact geometry. Invariants of compact manifolds, such as Chern classes,

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have been generalised to singular spaces (MacPherson) and have got large developments.

Let us turn our attention to a more global object, a polynomial function $f : \mathbb{C}^n \rightarrow \mathbb{C}$. The study of the singular fibration produced by a polynomial in the affine space was pioneered by Broughton [Br] more than 20 years ago, and took a certain amplex ever since. In the same time Pham [Ph] was interested in regularity conditions under which a polynomial has good behaviour at infinity, and Hà and Lê [HL] proved a criterion for detecting “atypical values” (see below) in two complex variables.

One may raise many natural questions in analogy to the ones which have been treated in the local setting of holomorphic functions. The first classification results in the global affine setting and the first lists of families of polynomials [SS, Ar] showed new phenomena and gave rise to more results and open questions.

Take the following example, which was given by Broughton [Br].

EXAMPLE 1.1. $f : \mathbb{C}^2 \rightarrow \mathbb{C}$, $f(x, y) = x + x^2y$.¹

One checks that f has no singular points. We have $f^{-1}(\varepsilon) = \{y = (\varepsilon - x)/x^2\}$ for $\varepsilon \neq 0$ and $f^{-1}(0) = \{x(xy + 1) = 0\}$. Therefore $f^{-1}(\varepsilon)$ is homeomorphic to $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, whereas $f^{-1}(0)$ is homeomorphic to the disjoint union $\mathbb{C} \sqcup \mathbb{C}^*$. (In order to get some more grip on the picture, it is interesting to consider the same equations over the reals and to draw the curves; what happens when $\varepsilon \rightarrow 0$?).

In the local setting, the presence of an isolated singularity is the only reason for the non-triviality of the local fibration associated to a function germ g : the fibre containing the singularity is contractible but the nearby fibres are not (they have non-trivial homology). In the global setting, we have just seen that the fibres of a polynomial may not be all the same (up to homeomorphisms), *even without the presence of singularities*.

QUESTION 1.2. What are the “atypical values” of a polynomial function? What sort of “singularities” are the cause of such atypical values?

To state this more precisely, we introduce the following:

Definition 1.3. One says that f is *topologically trivial* at $t_0 \in \mathbb{C}$ if there is a neighbourhood D of $t_0 \in \mathbb{C}$ such that the restriction $f|_D : f^{-1}(D) \rightarrow D$ is a topologically trivial fibration. If t_0 does not satisfy this property, then we say that t_0 is an *atypical value* and that $f^{-1}(t_0)$ is an *atypical fibre*. We shall denote by $\text{Atyp} f$ the set of atypical fibres of f .

¹The degree 3 plane curve $f = 0$ also appears in Newton’s manuscripts.

In the above example, we have seen that the set $\text{Atyp} f$ contains the value 0. One may show that this inclusion is an equality (exercise).

There is actually the following general inclusion:

$$f(\text{Sing} f) \subset \text{Atyp} f,$$

where $\text{Sing} f := Z(\text{Jac} f)$ denotes the singular locus of f , without any condition on its dimension.

Since $\text{Sing} f$ is an algebraic set and since the set of critical values $f(\text{Sing} f)$ is discrete in $\mathbb{C}$, by the Tarski-Seidenberg theorem $f(\text{Sing} f)$ is semi-algebraic hence finite. It turns out that the set of atypical values $\text{Atyp} f$ is also *finite*. The proof of the finiteness of the set of atypical values has been sketched by Thom [Th] and uses stratifications. A complete proof along this lines can be deduced from Verdier's study on Bertini-Sard theorems [Ve]. Another proof using the resolution of singularities was given by Pham [Ph, Appendix].

We shall see that in 2 variables one has a very precise meaning of *singularities at infinity*². This is not the case in higher dimensions: the problem of characterising the atypical non-singular values is still open!

Another evidence for the impact of singularities at infinity is the following famous open problem:³

CONJECTURE 1.4 (Jacobian Conjecture). Let $f, h \in \mathbb{C}[x, y]$. If $Z(\text{Jac}(f, h)) = \emptyset$ then f is equivalent to x modulo an automorphism of $\mathbb{C}^2$.

Indeed, this conjecture has the following equivalent formulation (cf. [LW] and [ST]) in terms of singularities at infinity:

CONJECTURE 1.5. If $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ has no critical points but has singularities at infinity then, for any polynomial $h : \mathbb{C}^2 \rightarrow \mathbb{C}$, the critical locus $Z(\text{Jac}(f, h))$ is not empty.

Proof. If the polynomial f has no critical points and no singularities at infinity then all the fibres of f are CW-complexes with trivial homotopy groups, hence contractible (see later). In this case the Abhyankar-Moh theorem tells that f is linearisable. The not covered case is therefore the one of singularities at infinity. $\square$

After this short motivation, we want to address the following questions in the setting of complex maps $F : \mathbb{C}^n \rightarrow \mathbb{C}^k$ and for some aspects in the setting of real maps $G : \mathbb{R}^n \rightarrow \mathbb{R}^k$ and also in the more recently emerging setting of “mixed” polynomials:

²there are actually over 10 characterisations of singularities at infinity in 2 variables, cf. [Du, Ti1]

³A similar conjecture holds in $\mathbb{C}^n$.

1. Bifurcation locus of polynomial maps. Regularity conditions at infinity (Malgrange, Łojasiewicz, Rabier, Gaffney, Jelonek etc).
2. How to detect atypical values.
3. Numerical invariants and how to compute them.
4. Newton polyhedra.
5. Mixed polynomials.
6. Deformations of polynomials. The singularity exchange at infinity.

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