

Syllabus: Python-illustrated combinatorial functional analysis

Instructor Information

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Course Description

The main objects of study in this course will be certain collections of finite subsets of that natural numbers called *regular families* and the norms and Banach spaces associated with families. Our starting example of a regular family of finite subsets are the Schreier sets defined as follows:

$$\mathcal{S} = \{F \subset \mathbb{N} : |F| \leq \min F\}.$$

Here $|F|$ is the cardinality of the set F . These sets were introduced by Józef Schreier in 1929. Schreier was a contemporary of Banach and contributing member Polish school of mathematics in Lwów, which developed many of the foundational mathematical theorems in functional analysis, set theory, and topology.

Schreier introduced these sets to solve a problem of Banach and Saks which asked for the complexity of the convergence of convex combinations of weakly null sequences in a Banach space. The Schreier sets are now properly understood as trees of subsets of natural numbers which are hereditary, spreading, and compact in the topology the product topology on $2^{\mathbb{N}}$. Any such collection of sets is called a regular family of subsets of $\mathbb{N}$. In modern functional analysis and combinatorics, regular families of subsets of $\mathbb{N}$ have proven to be an extremely useful tool.

In Banach space theory, for each regular family $\mathcal{F}$ there is a separable Banach space $X_{\mathcal{F}}$ whose norm satisfies

$$\|(a_i)\|_{\mathcal{F}} = \sup_{F \in \mathcal{F}} \sum_{i \in F} |a_i|$$

for all eventually null sequences of real numbers (a_i) . These are members of a class of Banach spaces called *combinatorial Banach spaces*. In this course we will study the geometry of these Banach spaces and the bounded linear operators defined on them.

Also in this course, provide many applications of regular families to many disparate areas of mathematics including topology, combinatorics, and number theory. Many of these connections have been discovered by using Python code to count or generate lists of such sets. In this course we will study and write our own code in hope of making further predictions.

Some of the material we cover will be classical but we will also emphasize many recent results in this area and develop new lines of research.

Textbook

No textbook is required. Lecture notes will be prepared. All other material will come from papers and other notes freely available online.

Prerequisites

Students should have had a first course in functional analysis and basic familiarity with python. Although we will be working with Banach spaces, it is not necessary to have familiarity with techniques in Banach space theory. These will be introduced as necessary.

Mathematical Course Topics

1. Schreier's space X_S and its isomorphic properties.
2. Defining the set $\mathcal{S}_\alpha$ for each countable ordinal α .
3. Tree properties of regular families and the Cantor Bendixon index.
4. Repeated averages and weakly null sequences.
5. Combinatorial spaces are c_0 saturated.
6. Surjective isometries on combinatorial spaces and their duals.
7. Schreier sets and their connections to the Fibonacci sequence and other recursions.
8. Polyhedral norms and combinatorial spaces.
9. Extreme points of combinatorial spaces and the λ -property.
10. Tsirelson's space and its variants.

Python Coding

In this course, we will use Python to write code to compute the elements of various regular families and the associated norms. This code will help us to compute and make predictions regarding the structure of the extreme points and recursions.

Projects

There are several open problems at the intersection of combinatorics and Banach space theory that we will explore in this class. We will investigate some of these directions and very likely make some progress.

Grading

Grades are based on the completion of homework sets and the presentation of results.

Homework and Projects

Look at Ramsey Methods in Analysis for some notes. Look at the Handbook article for some notes.

1. Introduce the finite order sets $\mathcal{A}_n$. Show that $\mathcal{A}_n$ the space X_n is isomorphic to c_0 . This is trivial. Show the basis is unconditional. Give examples of conditional basis for c_0 . Look at Ryan's notes.

In https://eprints.lancs.ac.uk/id/eprint/39691/4/BirdLaustsen_sdlatex.pdf we have a proof that Schreier space does not embed implies that the basis is shrinking since the basis is unconditional

2. Prove Theorem 3.3.1 that if (u_n) is unconditional TFAE X contains ℓ_1 (complementably). This will show Schreier space has a shrinking basis. As an exercise show with the embedding that the basis is shrinking. Also characterize Shrinking bases like in my paper with Dan.
4. The same hold for c_0 and boundedly complete.
5. Characterize the extreme points of X_n and characterize the isometries. This is interesting.
6. State and prove Mazur's theorem that each norm convergent sequence in X has convex combinations that converge weakly not 0. <http://mathonline.wikidot.com/mazur-s-theorem>
7. Prove James' Theorem that c_0 are ℓ_1 are not distortable.
8. Introduce the Schreier space and prove the following properties:
 - (a) The unit vector basis is weakly null.
 - (b) Explain how X_{S_1} can be embedded in the $C(S_1)$.
 - (c) Compute the Cantor-Bendixon index on S_1 .
 - (d) Explain the dual of Schreier space and the extreme points of Schreier spaces. Explain how they are hard to characterize.
 - (e) Have the write a python codes to produce all Schreier sets up to a certain cardinality.
 - (f) Mention the modifications of Schreier sets due to Hung and I and have them proved the connection.
 - (g) Can we say anything interesting about the space corresponding to these sets. What are the extreme points? We know the isometries.
 - (h) Prove that Schreier space contains c_0 but does not contain ℓ_p .
 - (i) What about a family that is the union of 2 Schreier sets. I may need to introduce the composition notations.